Multidimensional Scaling-Based TDOA Localization Scheme Using an Auxiliary Line

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***Abstract*—This work deals with source localization with time-difference-of-arrival (TDOA) measurements in two-dimensional (2-D) scenarios. Although the celebrated two-step weighted least squares (2WLS) method is quite successful, its drawback lies in an ill-conditioning problem when the sensor array is quasi-linear. This work presents a multidimensional scaling (MDS)-based localization scheme. Based on the subspace analysis of the scalar product matrix, an auxiliary line is defined in the plane, close to the global minimizer of the cost function. Then, the minimizer on the auxiliary line is found as the estimation of the source position. Simulations show that the proposed scheme achieves high localization accuracy for all kinds of sensor arrays including quasi-linear arrays.**

***Index Terms*—Multidimensional scaling (MDS), source localization, subspace, time-difference-of-arrival (TDOA).**

I. Introduction

FINDING the position of a single passive source using time-difference-of-arrival (TDOA) measurements from an array of spatially separated sensors at known locations has been an important problem in radar, sonar, mobile communications, multimedia, and wireless sensor networks [1]. The localization problem is usually converted into the minimization problem of cost functions [2]–[4]. Due to its accuracy and computational efficiency, the two-step weighted least squares (2WLS) method [5]–[7] minimizing the spherical least squares error function is the most widely used method. However, 2WLS method faces an ill-conditioning problem that the measurement matrix to be inverted will become ill-conditioned when the array is quasi-linear, resulting in large estimation error [6].

Recently, a new cost function for TDOA localization has been introduced, defined as the norm of the difference matrix between two scalar product matrices in the multidimensional scaling (MDS) framework [8]. The first application of the MDS framework to localization problems is for time-of-arrival (TOA) measurements [9], [10]. It is extended to TDOA cases in [8]. However, the solution to the minimization problem of the cost function in [8] again needs to invert a measurement matrix and has the same ill-conditioning problem as 2WLS method does.

In this work, we present a new solution to the minimization problem of the MDS-based cost function. First, we develop the subspace analysis of the scalar product matrix. Based on the subspace analysis, we define an auxiliary line in the plane that is close to the global minimizer of the cost function. Then, we find the minimizer of the cost function on the auxiliary line and define it as the estimation of the source position. To describe the performance of the proposed method, we divide the category of arbitrary arrays [5] into two subcategories: quasi-linear arrays and normal arrays. Simulations show that the estimation error of the proposed method is slightly larger than that of 2WLS method for normal arrays, equal to 2WLS method for linear arrays, and significantly smaller than 2WLS method for quasi-linear arrays.

The notations used in this letter are defined as follows. Boldface lowercase letter $\boldsymbol{a}$ and boldface uppercase letter $\mathbf{A}$ represent vector and matrix, respectively. $\mathbf{1}_N$ and $\mathbf{0}_N$ stand for N dimensional column vectors of all ones and all zeros, respectively. $\operatorname{diag}(a_1, \ldots, a_N)$ stands for diagonal matrix whose entries are $a_1, \ldots, a_N$. $\mathbf{I}_N$ stands for $N \times N$ identity matrix. $\dot{\mathbf{A}}$ and $\ddot{\mathbf{A}}$ stand for first- and second-order derivatives of $\mathbf{A}$ with respect to t, respectively. $\mathrm{E}\{\cdot\}$ is the expectation operator. $\|\cdot\|$ stands for Euclidean norm of a vector. $\odot$ stands for Schur product. Finally, $\mathrm{r}(\cdot)$, $\mathrm{tr}(\cdot)$, $(\cdot)^{\mathrm{T}}$, and $\|\cdot\|_{\mathrm{F}}$ stand for rank, trace, transpose, and Frobenius norm of a matrix, respectively.

II. System Model

Consider an array of $M \geq 5$ sensors and a single source in a two-dimensional (2-D) plane. Denote the known position of the *m*th sensor by $\mathbf{u}_m = [x_m, y_m]^{\mathrm{T}}$, $m = 1, \ldots, M$. Assign the first sensor as the reference. Denote the unknown position of the source by $\mathbf{u} = [x, y]^{\mathrm{T}}$, whose true value is $\mathbf{u}_0 = [x_0, y_0]^{\mathrm{T}}$. Other definitions are shown in Table I. Note that d_m, d_{m1}, $\mathbf{d}$, $\mathbf{Z}$, and $\mathbf{B}$ are functions of $\mathbf{u}$. When $\mathbf{u} = \mathbf{u}_0$, they reach their true values.

By multiplying the signal propagation speed, the TDOA measurement converts to the range difference measurement $\hat{d}_{m1}$, which is modeled as $\hat{d}_{m1} = d_{m1}^0 + q_m$, $m = 1, \ldots, M$, where q_m, $m = 2, \ldots, M$ is the measurement noise of the range difference, and $q_1 = 0$. Assume $\mathrm{E}\{q_m\} = 0$, $m = 2, \ldots, M$.

Since $\mathbf{B} = \mathbf{Z}^{\mathrm{T}}\mathbf{D}\mathbf{Z}$, the (*i,j*)th entry of $\mathbf{B}$ can be expressed as

$$[\mathbf{B}]_{i,j} = \frac{1}{2}(d_{i1} - d_{j1})^2 - \frac{1}{2}\left[(x_i - x_j)^2 + (y_i - y_j)^2\right], \quad 1 \leq i, j \leq M. \tag{1}$$

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TABLE I
DEFINITIONS OF SYMBOLS

Symbol	Definition	Description	True value
$d_m \in \mathbb{R}$	$\lVert \mathbf{u}_m - \mathbf{u} \rVert$	Range	d_m^0
$d_{m1} \in \mathbb{R}$	$d_m - d_1$	Range difference	d_{m1}^0
$\mathbf{d} \in \mathbb{R}^M$	$\left[d_1, \ldots, d_M\right]^{\mathrm{T}}$	Range vector	$\mathbf{d}_0$
$\mathbf{x} \in \mathbb{R}^M$	$\left[x_1, \ldots, x_M\right]^{\mathrm{T}}$	*x*-coordinate vector of sensors	–
$\mathbf{y} \in \mathbb{R}^M$	$\left[y_1, \ldots, y_M\right]^{\mathrm{T}}$	*y*-coordinate vector of sensors	–
$\mathbf{D} \in \mathbb{R}^{3\times 3}$	$\operatorname{diag}(1,1,-1)$	–	–
$\mathbf{Z} \in \mathbb{R}^{3\times M}$	$\left[\mathbf{x}-x\mathbf{1}_M, \mathbf{y}-y\mathbf{1}_M, \mathbf{d}\right]^{\mathrm{T}}$	Position coordinates matrix	$\mathbf{Z}_0$
$\mathbf{B} \in \mathbb{R}^{M\times M}$	$\mathbf{Z}^{\mathrm{T}}\mathbf{D}\mathbf{Z}$	Scalar product matrix	$\mathbf{B}_0$
$\hat{d}_{m1} \in \mathbb{R}$	$d_{m1}^0 + q_m$	Range difference measurement	–
$\hat{\mathbf{d}}_d \in \mathbb{R}^M$	$\left[\hat{d}_{11}, \ldots, \hat{d}_{M1}\right]^{\mathrm{T}}$	Vector of range difference measurements	–
$\mathbf{q} \in \mathbb{R}^M$	$\left[q_1, \ldots, q_M\right]^{\mathrm{T}}$	Vector of range difference measurement noises	–
$\mathbf{Z}_1 \in \mathbb{R}^{3\times M}$	$\left[\mathbf{x}, \mathbf{y}, \hat{\mathbf{d}}_d\right]^{\mathrm{T}}$	–	–

If we substitute $\hat{d}_{m1}$ for d_{m1} in (1) $1 \le m \le M$, we get the noisy scalar product matrix $\hat{\mathbf{B}} \in \mathbb{R}^{M\times M}$, whose (*i,j*)th entry is

$$\left[\hat{\mathbf{B}}\right]_{i,j} = \frac{1}{2}\left(\hat{d}_{i1} - \hat{d}_{j1}\right)^2 - \frac{1}{2}\left[(x_i - x_j)^2 + (y_i - y_j)^2\right], \quad 1 \le i, j \le M. \tag{2}$$

Note that $\mathbf{B}$ and $\hat{\mathbf{B}}$ are symmetric matrices. $\mathbf{B}$ is a function matrix of $\mathbf{u}$, whereas $\hat{\mathbf{B}}$ is a constant matrix independent of $\mathbf{u}$. The cost function with respect to $\mathbf{u}$ is defined as

$$f(\mathbf{u}) = \left\lVert \mathbf{B} - \hat{\mathbf{B}} \right\rVert_{\mathrm{F}}^2. \tag{3}$$

The $\hat{\mathbf{u}}_0 = [\hat{x}_0, \hat{y}_0]^{\mathrm{T}}$ which minimizes (3) is defined as the estimation of the source position.

III. Subspace Analysis

This section presents the subspace analysis of $\hat{\mathbf{B}}$. Denote

$$\mathbf{Z}_2 = [\mathbf{x} - x_0\mathbf{1}_M, \mathbf{y} - y_0\mathbf{1}_M, \mathbf{d}_0 + \mathbf{q}, 0.5\mathbf{q} \odot \mathbf{q} + \mathbf{d}_0 \odot \mathbf{q}, \mathbf{1}_M]^{\mathrm{T}} \in \mathbb{R}^{5\times M},\ \mathbf{Z}_3 = [\mathbf{x} - x_0\mathbf{1}_M, \mathbf{d}_0 + \mathbf{q}, 0.5\mathbf{q} \odot \mathbf{q} + \mathbf{d}_0 \odot \mathbf{q}, \mathbf{1}_M]^{\mathrm{T}} \in \mathbb{R}^{4\times M},$$

$$\mathbf{D}_2 = \operatorname{diag}\left(1, 1, -1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \in \mathbb{R}^{5\times 5},$$

$$\mathbf{D}_3 = \begin{bmatrix} 1 + a^2 & 0 & 0 & ab \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ ab & 0 & 1 & b^2 \end{bmatrix} \in \mathbb{R}^{4\times 4}, \text{ where } a, b \in \mathbb{R}.$$

Property 1: $r\left(\hat{\mathbf{B}}\right) \le 5$. If sensors are collinear, then $\mathrm{r}\left(\hat{\mathbf{B}}\right) \le 4$.

Proof: It can be verified from (2) that

$$\hat{\mathbf{B}} = \mathbf{Z}_2^{\mathrm{T}}\mathbf{D}_2\mathbf{Z}_2. \tag{4}$$

As a consequence, $\mathrm{r}\left(\hat{\mathbf{B}}\right) \le \mathrm{r}(\mathbf{Z}_2) \le 5$.

If sensors are collinear, without loss of generality we assume $y_m - y = a(x_m - x) + b$, $a, b \in \mathbb{R}$, $m = 1, \ldots, M$. Then (4) becomes

$$\hat{\mathbf{B}} = \mathbf{Z}_3^{\mathrm{T}}\mathbf{D}_3\mathbf{Z}_3. \tag{5}$$

As a consequence, $\mathrm{r}\left(\hat{\mathbf{B}}\right) \le \mathrm{r}(\mathbf{Z}_3) \le 4$. ■

Property 2. If $r\left(\hat{\mathbf{B}}\right) = 5$, or if sensors are collinear and $r\left(\hat{\mathbf{B}}\right) = 4$, then $\hat{\mathbf{B}}\mathbf{v} = 0$ implies $\mathbf{Z}_1\mathbf{v} = 0$ and $\mathbf{1}_M^{\mathrm{T}}\mathbf{v} = 0$, $\mathbf{v} \in \mathbb{R}^M$.

Proof: If $r\left(\hat{\mathbf{B}}\right) = 5$ and $\hat{\mathbf{B}}\mathbf{v} = 0$, denote $\mathbf{Z}_2\mathbf{v} = [l_1, \ldots, l_5]^{\mathrm{T}} \in \mathbb{R}^5$. According to (4), $5 = \mathrm{r}\left(\hat{\mathbf{B}}\right) \le \mathrm{r}\left(\mathbf{Z}_2^{\mathrm{T}}\right) \le 5$. So $\mathrm{r}\left(\mathbf{Z}_2^{\mathrm{T}}\right) = 5$. It follows from (4) and $\hat{\mathbf{B}}\mathbf{v} = 0$ that $\mathbf{Z}_2^{\mathrm{T}}[l_1, l_2, -l_3, l_5, l_4]^{\mathrm{T}} = 0$. Since $\mathrm{r}\left(\mathbf{Z}_2^{\mathrm{T}}\right) = 5$, $[l_1, l_2, -l_3, l_5, l_4]^{\mathrm{T}} = 0$, so $\mathbf{Z}_2\mathbf{v} = 0$. As $\hat{\mathbf{d}}_d = \mathbf{d}_0 - d_1^0\mathbf{1}_M + \mathbf{q}$ is a linear combination of $\mathbf{d}_0 + \mathbf{q}$ and $\mathbf{1}_M$, we have $\mathbf{Z}_1\mathbf{v} = 0$ and $\mathbf{1}_M^{\mathrm{T}}\mathbf{v} = 0$.

If sensors are collinear and $r\left(\hat{\mathbf{B}}\right) = 4$ and $\hat{\mathbf{B}}\mathbf{v} = 0$, denote $\mathbf{Z}_3\mathbf{v} = [l_1, \ldots, l_4]^{\mathrm{T}} \in \mathbb{R}^4$. Following the same procedure as above, we have $\left[\left(1 + a^2\right)l_1 + abl_4, -l_2, l_4, abl_1 + l_3 + b^2l_4\right]^{\mathrm{T}} = 0$. So $\mathbf{Z}_3\mathbf{v} = 0$, implying $\mathbf{Z}_1\mathbf{v} = 0$ and $\mathbf{1}_M^{\mathrm{T}}\mathbf{v} = 0$. ■

Remark: Besides the two cases in Property 2, there are other cases, i.e., the case when $r\left(\hat{\mathbf{B}}\right) = 4$ but sensors are not collinear, and the case when $r\left(\hat{\mathbf{B}}\right) \le 3$. These cases will happen on either of the following conditions: 1) $\mathrm{E}\left\{q_2^2\right\} = \cdots = \mathrm{E}\left\{q_M^2\right\} = 0$, so $0.5\mathbf{q} \odot \mathbf{q} + \mathbf{d}_0 \odot \mathbf{q} = 0$; 2) $q_2, \ldots, q_M$ are simultaneously appointed to particular values so that $\mathbf{d}_0 + \mathbf{q}$ or $0.5\mathbf{q} \odot \mathbf{q} + \mathbf{d}_0 \odot \mathbf{q}$ is a linear combination of $\mathbf{x}, \mathbf{y}, \mathbf{1}_M$. For simplicity, we assume that condition 1) is not satisfied. Besides, since $q_2, \ldots, q_M$ are random variables, it is reasonable to assume that condition 2) will not be satisfied in practice, either. As a result, the two cases in Property 2 are the only cases that will happen in practice. In other words, $4 \le \mathrm{r}\left(\hat{\mathbf{B}}\right) \le 5$, and the sufficient and necessary condition of $\mathrm{r}\left(\hat{\mathbf{B}}\right) = 4$ is that sensors are collinear.

IV. Localization Scheme Design

A. Defining the Auxiliary Line

We use $\mathbf{Z}_1\mathbf{v}/\mathbf{1}_M^{\mathrm{T}}\mathbf{v} = [x_v, y_v, -d_1^v]^{\mathrm{T}} \in \mathbb{R}^3$, $\mathbf{v} \in \mathbb{R}^M$ to estimate $\left[x_0, y_0, -d_1^0\right]^{\mathrm{T}}$. The difference vector $\mathbf{e}_1$ between them is $\mathbf{e}_1 = [\mathbf{x} - x_0\mathbf{1}_M, \mathbf{y} - y_0\mathbf{1}_M, \mathbf{d}_0 + \mathbf{q}]^{\mathrm{T}}\mathbf{v}/\mathbf{1}_M^{\mathrm{T}}\mathbf{v} \in \mathbb{R}^3$. Define $\mathbf{e}_2 = \mathbf{Z}_0^{\mathrm{T}}\mathbf{D} \cdot \mathbf{e}_1 = \left(\mathbf{B}_0 - \mathbf{d}_0\mathbf{q}^{\mathrm{T}}\right)\mathbf{v}/\mathbf{1}_M^{\mathrm{T}}\mathbf{v} \in \mathbb{R}^M$. Since $\hat{\mathbf{B}}$ is the approximation of $\mathbf{B}_0$, and $\mathrm{E}\left\{\mathbf{d}_0\mathbf{q}^{\mathrm{T}}\right\} = 0$, we can use $\mathbf{e}_3 = \hat{\mathbf{B}}\mathbf{v}/\mathbf{1}_M^{\mathrm{T}}\mathbf{v} \in \mathbb{R}^M$ to be the approximation of $\mathbf{e}_2$. If

$\mathbf{v}$ is properly chosen so that the elements of $\mathbf{e}_1$ are small, then the elements of $\mathbf{e}_2$ and $\mathbf{e}_3$ should be relatively small. As a result, $\|\mathbf{e}_3\|$ should be relatively small. The term *relatively small* means that the current quantity is smaller than most of the quantities determined by all possible choices of $\mathbf{v}$.

Next, we give a detailed description of $\|\mathbf{e}_3\|$. Since $\hat{\mathbf{B}}$ is symmetric, suppose the eigenvalue decomposition of $\hat{\mathbf{B}}$ is

$$\hat{\mathbf{B}} = [\mathbf{v}_1, \ldots, \mathbf{v}_M] \operatorname{diag}(s_1, \ldots, s_M) [\mathbf{v}_1, \ldots, \mathbf{v}_M]^{\mathrm{T}}, \tag{6}$$

where $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M \in \mathbb{R}^M$ are orthonormal vectors, $s_1, \ldots, s_M \in \mathbb{R}$, and $|s_1| \geq \ldots \geq |s_M|$. Then, $\mathbf{v}$ can be expressed as $\mathbf{v} = k_1\mathbf{v}_1 + \cdots + k_M\mathbf{v}_M$, $k_1, \ldots, k_M \in \mathbb{R}$. Denote $a_m = \mathbf{1}_M^{\mathrm{T}}\mathbf{v}_m \in \mathbb{R}$, $m = 1, \ldots, M$. According to Properties 1 and 2, $s_m = 0$, $\mathbf{Z}_1\mathbf{v}_m = 0$, and $a_m = 0$ for $m = 6, \ldots, M$. Thus, $\mathbf{Z}_1\mathbf{v}/\mathbf{1}_M^{\mathrm{T}}\mathbf{v}$ is determined by $k_1, \ldots, k_5$. Since $\mathbf{e}_3 = \hat{\mathbf{B}}\mathbf{v}/\mathbf{1}_M^{\mathrm{T}}\mathbf{v}$ is a homogeneous function of $\mathbf{v}$ of degree zero, we can assume $\mathbf{1}_M^{\mathrm{T}}\mathbf{v} = 1$. Thus

$$\begin{aligned} \|\mathbf{e}_3\|^2 &= \left\|\hat{\mathbf{B}}\mathbf{v}\right\|^2 = \|k_1 s_1 \mathbf{v}_1 + \cdots + k_5 s_5 \mathbf{v}_5\|^2 \\ &= k_1^2 s_1^2 + \cdots + k_5^2 s_5^2, \\ &\text{s.t. } k_1 a_1 + \cdots + k_5 a_5 = 1. \end{aligned} \tag{7}$$

Next, we deduce a necessary condition that $\|\mathbf{e}_3\|$ is relatively small. Suppose the eigenvalues of $\mathbf{B}_0$ are $\sigma_1, \ldots, \sigma_M \in \mathbb{R}$, which are ordered so that $\sum (s_i - \sigma_i)^2$ is minimized. Then, according to Wielandt–Hoffman Theorem [11]

$$\sum (s_i - \sigma_i)^2 \leq \left\|\hat{\mathbf{B}} - \mathbf{B}_0\right\|_{\mathrm{F}}^2. \tag{8}$$

It can be verified from (1) and (2) that

$$\begin{aligned} \hat{\mathbf{B}} - \mathbf{B}_0 = {} & (\mathbf{d}_0 \odot \mathbf{q}) \cdot \mathbf{1}_M^{\mathrm{T}} + \mathbf{1}_M \cdot (\mathbf{d}_0 \odot \mathbf{q})^{\mathrm{T}} - \mathbf{d}_0\mathbf{q}^{\mathrm{T}} - \mathbf{q}\mathbf{d}_0^{\mathrm{T}} \\ & + 0.5 (\mathbf{q} \odot \mathbf{q}) \cdot \mathbf{1}_M^{\mathrm{T}} + 0.5\mathbf{1}_M \cdot (\mathbf{q} \odot \mathbf{q})^{\mathrm{T}} - \mathbf{q}\mathbf{q}^{\mathrm{T}}. \end{aligned} \tag{9}$$

According to (1) and (9), all of the elements of $\hat{\mathbf{B}} - \mathbf{B}_0$ are products of $\mathbf{q}$, whereas none of the elements of $\mathbf{B}_0$ depend on $\mathbf{q}$. As long as $\|\mathbf{q}\|$ is small enough, the sum of the squared elements of $\hat{\mathbf{B}} - \mathbf{B}_0$ will be much smaller than that of $\mathbf{B}_0$. Thus

$$\left\|\hat{\mathbf{B}} - \mathbf{B}_0\right\|_{\mathrm{F}}^2 \ll \|\mathbf{B}_0\|_{\mathrm{F}}^2 = \sum \sigma_i^2. \tag{10}$$

It follows from (8) and (10) that $\sum (s_i - \sigma_i)^2 \ll \sum \sigma_i^2$. Rearrange σ_i^2 so that $\sigma_{i_1}^2 \geq \cdots \geq \sigma_{i_M}^2$. Since $\mathrm{r}(\mathbf{B}_0) \leq 3$, $\sigma_{i_k}^2 = 0$ for $k = 4, \ldots, M$. It can be verified that in most cases $\sigma_{i_3}^2 \gg 0$. (The exception is when the sensors and the source simultaneously lie on the same line. In this case, there are infinite points satisfying the TDOA conditions.) As a result, in most cases s_1^2, s_2^2, s_3^2 are much larger than s_4^2, s_5^2. According to (7), if $\|\mathbf{e}_3\|$ is relatively small, then k_1^2, k_2^2, k_3^2 should be close to 0. Denote $[x_v, y_v]^{\mathrm{T}}$ by $[x_v^0, y_v^0]^{\mathrm{T}}$ when $k_1 = k_2 = k_3 = 0$. Then

$$[x_v^0, y_v^0]^{\mathrm{T}} = k_4[\mathbf{x}, \mathbf{y}]^{\mathrm{T}}\mathbf{v}_4 + k_5[\mathbf{x}, \mathbf{y}]^{\mathrm{T}}\mathbf{v}_5, \text{ s.t. } k_4 a_4 + k_5 a_5 = 1. \tag{11}$$

We conclude that, if $\mathbf{v}$ is properly chosen so that $\|\mathbf{e}_3\|$ is relatively small, then $[x_v, y_v]^{\mathrm{T}}$ should be relatively close to the trail of $[x_v^0, y_v^0]^{\mathrm{T}}$ determined by (11).

Now, we define the auxiliary line according to (11). If sensors are not collinear, the trail of $[x_v^0, y_v^0]^{\mathrm{T}}$ is a line that connects $[\mathbf{x}, \mathbf{y}]^{\mathrm{T}}\mathbf{v}_4/a_4 \in \mathbb{R}^2$ and $[\mathbf{x}, \mathbf{y}]^{\mathrm{T}}\mathbf{v}_5/a_5 \in \mathbb{R}^2$. This line is defined as the auxiliary line. Since the $\|\mathbf{e}_3\|$ corresponding to $\hat{\mathbf{u}}_0$ is relatively small, $\hat{\mathbf{u}}_0$ should be close to the auxiliary line. On the other hand, if sensors are collinear, according to Properties 1 and 2, $s_5 = 0$, $\mathbf{Z}_1\mathbf{v}_5 = 0$, $a_5 = 0$, so the trail of $[x_v^0, y_v^0]^{\mathrm{T}}$ degenerates into a single point of $[\mathbf{x}, \mathbf{y}]^{\mathrm{T}}\mathbf{v}_4/a_4$. Call the line that sensors lie on as L_1. It can be verified that $[\mathbf{x}, \mathbf{y}]^{\mathrm{T}}\mathbf{v}/\mathbf{1}_M^{\mathrm{T}}\mathbf{v} \in \mathbb{R}^2$ lies on L_1 for all $\mathbf{v} \in \mathbb{R}^M$. Since the $\|\mathbf{e}_3\|$ corresponding to $\mathbf{v} = \mathbf{v}_4$ is relatively small, $[\mathbf{x}, \mathbf{y}]^{\mathrm{T}}\mathbf{v}_4/a_4$ should be the point on L_1 that is relatively close to $\hat{\mathbf{u}}_0$ comparing to other points of $[\mathbf{x}, \mathbf{y}]^{\mathrm{T}}\mathbf{v}/\mathbf{1}_M^{\mathrm{T}}\mathbf{v}$ on L_1. As a consequence, $\hat{\mathbf{u}}_0$ should be close to the line called L_2 that goes through $[\mathbf{x}, \mathbf{y}]^{\mathrm{T}}\mathbf{v}_4/a_4$ and is perpendicular to L_1. We define L_2 as the auxiliary line.

In summary, the auxiliary line defined above is close to $\hat{\mathbf{u}}_0$. As a result, the point minimizing (3) on the auxiliary line will be a good approximation of $\hat{\mathbf{u}}_0$.

TABLE II
PROPOSED LOCALIZATION SCHEME

1: Calculate the eigenvalue decomposition of $\hat{\mathbf{B}}$ as (6), where $\hat{\mathbf{B}}$ is defined in (2). Calculate $a_m = \mathbf{1}_M^{\mathrm{T}}\mathbf{v}_m \in \mathbb{R}$, $m = 4, 5$.
2: Define $\mathbf{u}_0^{(1)} = [\mathbf{x}, \mathbf{y}]^{\mathrm{T}}\mathbf{v}_4/a_4 \in \mathbb{R}^2$ and $d_1^{(1)} = -\hat{\mathbf{d}}_d^{\mathrm{T}}\mathbf{v}_4/a_4 \in \mathbb{R}$.
3: If $a_5 \neq 0$, define $\mathbf{u}_f = [\mathbf{x}, \mathbf{y}]^{\mathrm{T}}\mathbf{v}_5/a_5$, $\mathbf{p} = \left(\mathbf{u}_f - \mathbf{u}_0^{(1)}\right) / \left\|\mathbf{u}_f - \mathbf{u}_0^{(1)}\right\|$; else, define $\mathbf{p} = [-(y_2 - y_1), x_2 - x_1]^{\mathrm{T}} / \|\mathbf{u}_2 - \mathbf{u}_1\|$, where $\mathbf{u}_f, \mathbf{p} \in \mathbb{R}^2$. Then the points on the auxiliary line are $\mathbf{u} = \mathbf{u}_0^{(1)} + t\mathbf{p} \in \mathbb{R}^2$, $t \in \mathbb{R}$.
4: Define $c_1 = \mathbf{p}^{\mathrm{T}}\left(\mathbf{u}_0^{(1)} - \mathbf{u}_1\right)$, $c_0 = \left\|\mathbf{u}_0^{(1)} - \mathbf{u}_1\right\|^2 - \left(d_1^{(1)}\right)^2$, $\Delta = c_1^2 - c_0$. If $\Delta > 0$, define $t_2 = -c_1 \pm \sqrt{\Delta}$; else, define $t_2 = -c_1$, where $c_1, c_0, \Delta, t_2 \in \mathbb{R}$. Define $\mathbf{u}_0^{(2)} = \mathbf{u}_0^{(1)} + t_2\mathbf{p} \in \mathbb{R}^2$.
5: Define $g(t) = 2\operatorname{tr}\left(\left(\mathbf{B} - \hat{\mathbf{B}}\right)\dot{\mathbf{B}}\right)$, $g'(t) = 2\operatorname{tr}\left(\left(\dot{\mathbf{B}}\right)^2 + \left(\mathbf{B} - \hat{\mathbf{B}}\right)\ddot{\mathbf{B}}\right)$, where $\mathbf{B}$ is calculated in Table I, and $\dot{\mathbf{B}}$ and $\ddot{\mathbf{B}}$ in Appendix. Use root-finding methods with $t = t_2$ as the initial value to find $t_3 \in \mathbb{R}$ so that $g(t_3) = 0$ and $g'(t_3) \geq 0$.
6: The estimation of the source position is $\mathbf{u}_0^{(3)} = \mathbf{u}_0^{(1)} + t_3\mathbf{p} \in \mathbb{R}^2$.

B. Proposed Localization Scheme

The proposed localization scheme is summarized in Table II. Note that (3) becomes a function of t when $\mathbf{u}$ is on the auxiliary line. So minimization of (3) converts to root-finding of $g(t)$, the derivative of (3) with respect to t.

Here, we explain the meaning of $\mathbf{u}_0^{(2)}$. Draw a circle whose center is $\mathbf{u}_1$ and radius $\left|d_1^{(1)}\right|$. $\mathbf{u}_0^{(2)}$ is the point on the auxiliary line which is closest to the circle, i.e., the intersections when the circle and the auxiliary line intersect, or the orthogonal projection from the center of the circle to the auxiliary line when they do not intersect. $\mathbf{u}_0^{(2)}$ is superior than $\mathbf{u}_0^{(1)}$ because that, when sensors are collinear, $\mathbf{u}_0^{(1)}$ will be confined on L_1 which is far away from $\hat{\mathbf{u}}_0$, whereas $\mathbf{u}_0^{(2)}$ will still be close to $\hat{\mathbf{u}}_0$.

V. Simulations

This section presents simulation results to demonstrate the performance of the proposed method by comparing it with 2WLS method.

Suppose that there are eight sensors and one source. The range difference measurements that are converted from TDOA measurements are generated by adding the zero-mean Gaussian noises to the true values. The covariance matrix of the noise [5] is $\mathbf{Q} = \mathrm{E}\left\{[q_2, \ldots, q_M]^{\mathrm{T}} \cdot [q_2, \ldots, q_M]\right\} = 0.5\sigma^2\left(\mathbf{I}_{M-1} + \mathbf{1}_{M-1}\mathbf{1}_{M-1}^{\mathrm{T}}\right)$. $q_1 = 0$.

We present three sets of experiments corresponding to normal, linear, and quasi-linear arrays, respectively. In Set-1, sensors and the source are uniformly distributed in the area of $[0, 100\,\mathrm{m}] \times [0, 100\,\mathrm{m}]$. In Set-2, sensors are on the line of $y = x$. The x coordinates of first and eighth sensors are 25 m and 75 m, respectively, whereas the x coordinates of the other sensors are uniformly distributed in the interval of (25 m, 75 m). The source is uniformly distributed in the area of $[0, 50\,\mathrm{m}] \times [50\,\mathrm{m}, 100\,\mathrm{m}]$. In Set-3, we have the same assumption as Set-2 with one change, adding a zero-mean Gaussian noise with variance $0.6^2\mathrm{m}^2$ to the y coordinate of each sensor. Let $\sigma^2 = \left(2 \times 0.3^2\right)\mathrm{m}^2$.

We compare the estimation errors of three methods. The first method is 2WLS method. We use 2WLS method for arbitrary arrays in Set-1 and that for linear arrays in Set-2 [5]. In Set-3, define $\mathbf{A} = [\mathbf{x}, \mathbf{1}_M]$ and $h = \left\|\mathbf{y} - \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{y}\right\|$. If $h > h_0$, we use 2WLS method for arbitrary arrays; otherwise, we use that for linear arrays, where h_0 is the preset threshold. We present four thresholds as $h_0 \in \{0, 1, 2, \infty\}$ (unit: m). The second method is exhaustive search to find the global minimizer of (3). The third method is the proposed method, where we use $10^{-10}\,\mathrm{m}$ as the threshold to decide whether $a_5 = 0$ in Step 3 of Table II. In each set of experiments, we conduct 10 000 experiments and record the estimation errors of the three methods. The estimation error is defined as $\|\hat{\mathbf{u}} - \mathbf{u}_0\|$, where $\hat{\mathbf{u}}$ is the estimation. We present the cumulative distribution function (CDF) of the estimation errors for comparison. Note that in Set-1 the probability of the sensor array being linear or quasi-linear is very small, so we can consider the CDF in Set-1 as the performance for normal arrays.

Fig. 1 shows the CDF of the three methods in Set-1 (normal arrays) and Set-2 (linear arrays). Fig. 2 shows that in Set-3 (quasi-linear arrays). We have the following conclusions. 1) The estimation error of the proposed method is slightly larger than that of 2WLS method for normal arrays, equal to 2WLS method for linear arrays, and significantly smaller than 2WLS method for quasi-linear arrays. Moreover, the CDF of the proposed method for quasi-linear arrays is nearly the same as that for linear arrays. This shows that the proposed method successfully avoids the ill-conditioning problem. 2) The CDF of the proposed method is close to that of exhaustive search, which shows that the minimizer of (3) on the auxiliary line is a good approximation of the global minimizer in the plane. 3) For normal arrays, even the estimation error of exhaustive search is slightly larger than that of 2WLS method. This shows that the cost function (3) needs refinement to improve the localization accuracy, which remains to be studied.

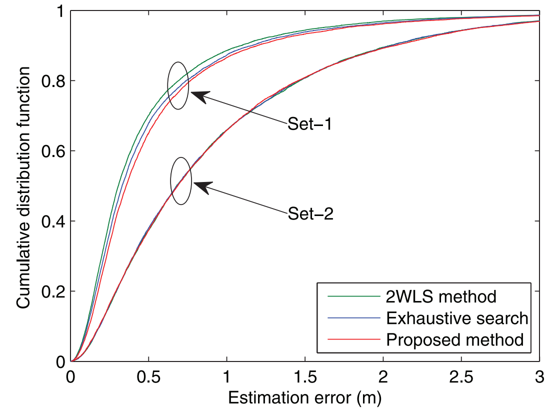

Fig. 1. CDFs of 2WLS method, exhaustive search, and the proposed method in Set-1 (normal arrays) and Set-2 (linear arrays) of experiments.

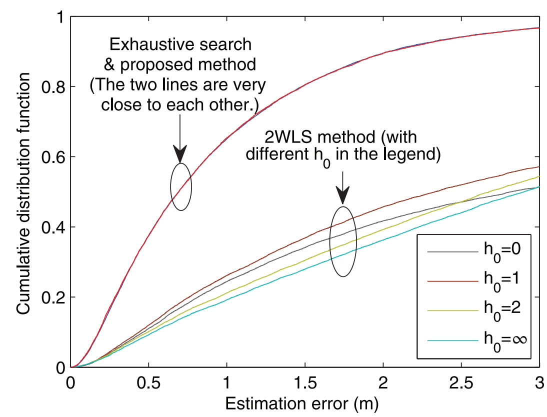

Fig. 2. CDFs of 2WLS method with $h_0 \in \{0, 1, 2, \infty\}$ (unit: m), exhaustive search, and the proposed method in Set-3 (quasi-linear arrays) of experiments.

VI. Conclusion

This work presents an MDS-based TDOA localization scheme using an auxiliary line, where the minimizer of the cost function on the auxiliary line is easy to find and is close to the global minimizer in the plane. Since the proposed scheme does not require matrix inversions, it avoids the ill-conditioning problem for quasi-linear sensor arrays. Future works may include refining the cost function to improve the localization accuracy, or extension to localization of multiple sources.

Appendix: Calculation of $\dot{\mathbf{B}}$ and $\ddot{\mathbf{B}}$

Denote $\mathbf{w}_m = \mathbf{u} - \mathbf{u}_m$, then $d_m = \|\mathbf{w}_m\|$, $\dot{d}_m = \mathbf{p}^{\mathrm{T}}\mathbf{w}_m/d_m$, $\ddot{d}_m = \left(1 - \dot{d}_m^2\right)/d_m$. $\dot{\mathbf{Z}} = \left[-\mathbf{1}_M\mathbf{p}^{\mathrm{T}}, \dot{\mathbf{d}}\right]^{\mathrm{T}}$, $\ddot{\mathbf{Z}} = \left[\mathbf{0}_M, \mathbf{0}_M, \ddot{\mathbf{d}}\right]^{\mathrm{T}}$. Finally, $\dot{\mathbf{B}} = \dot{\mathbf{Z}}^{\mathrm{T}}\mathbf{D}\mathbf{Z} + \mathbf{Z}^{\mathrm{T}}\mathbf{D}\dot{\mathbf{Z}}$, $\ddot{\mathbf{B}} = \ddot{\mathbf{Z}}^{\mathrm{T}}\mathbf{D}\mathbf{Z} + 2\dot{\mathbf{Z}}^{\mathrm{T}}\mathbf{D}\dot{\mathbf{Z}} + \mathbf{Z}^{\mathrm{T}}\mathbf{D}\ddot{\mathbf{Z}}$.

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