# DOA Estimation for Noncircular Sources with Multiple Noncoherent Subarrays

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***Abstract*— This letter addresses the direction-of-arrival estimation issue for noncircular sources with multiple noncoherent subarrays. First, we present the Cramer–Rao lower bound for this problem. Then, for multiple noncoherent subarrays, we propose the asymptotically minimum variance second-order estimators for the two cases of circular and noncircular sources, respectively. Furthermore, we propose a computationally efficient MUSIC-like algorithm for such arrays which can exploit the noncircularity of the sources. Finally, numerical examples have been provided to demonstrate the performance of the new methods.**

***Index Terms*— Noncircular signals, direction of arrival (DOA) estimation, MUSIC algorithm, time varying arrays, distributed sensor array networks.**

## I. Introduction

DIRECTION-OF-ARRIVAL (DOA) estimation of multiple sources using an array is an essential task in many applications, such as radar, sonar, communications, geophysics, tracking and localization [1]. In this work, we consider the DOA estimation problem using multiple noncoherent subarrays. Such arrays arise in the many important applications, such as time-varying arrays [5]–[11], subarray sampling [12], [13] and partially coherent arrays. It is also the case of distributed sensor array networks, in which each array is perfectly coherent locally but the correlation between different arrays is irrelevant [2]–[4].

All of the above works [2]–[13] focused on either deterministic or stochastic circular signals, while this work considers the case of noncircular signals. Noncircular signals are usually encountered in the context of communications, such as AM, MASK, BPSK or UQPSK signals. It has been shown that, the noncircularity of complex signals can be exploited to improve the performance of DOA estimation [14]–[19], [27], beamforming [20]–[22] and time delay estimation [23], [24].

Existing DOA estimation methods for noncircular signals (e.g., [14]–[19] and the references therein) commonly assume a single coherent array, and none of them applies to multiple noncoherent subarrays. The goal of this work is to develop DOA estimation methods for noncircular signals applicable to generalized multiple noncoherent subarrays.

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The main contributions of this work are as follows. First, we present the CRB of DOA estimation for noncircular complex Gaussian (NCG) sources with multiple noncoherent subarrays. Then, the GLS estimator [10] is improved to be an asymptotically minimum variance (AMV) second-order (SO) estimator for circular sources, and the AMV SO estimator for noncircular sources [15] is extended for multiple noncoherent subarrays. For multiple sources, these methods require multi-dimensional nonlinear optimization, which prompts us to design a computationally simple MUSIC-like method subsequently. Specifically, we propose a weighted MUSIC algorithm for noncircular sources, which fuses the spatial spectrum of the subarrays in a weighted summation manner.

*Notations:* $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and Hermitian transpose, respectively. $\delta_{l,k}$ is the Kronneck function. $||\cdot||_F^2$, $\Re(\cdot)$ and $\Im(\cdot)$ stand for the Frobenius norm, real and imaginary part operators, respectively. vec$(\cdot)$ is the "vectorization" operator stacking the columns of the matrix one below another, and v$(\cdot)$ denotes the operator obtained from vec$(\cdot)$ by eliminating all supradiagonal elements of the matrix. $\otimes$ denotes the Kronecker product. $\mathbf{I}_M$ stands for an $M \times M$ identity matrix.

## II. Problem Statement, ML Estimator and CRB

### A. Signal Model

Consider an array consisting of $L$ noncoherent subarrays receiving $d$ narrowband far-field signals impinging with unknown DOAs $\theta_1, \cdots, \theta_d$. The $l$-th subarray has $M_l$ sensors. The output of the $l$-th subarray at time $t_l$ at some local reference point can be expressed as

$$\mathbf{y}_l(t_l) = \mathbf{A}_l(\mathbf{\Theta})\mathbf{x}(t_l) + \mathbf{n}_l(t_l), \quad t_l = 1, \cdots, N_l \tag{1}$$

where $\mathbf{A}_l(\mathbf{\Theta}) = [\mathbf{a}_l(\theta_1) \ \cdots \ \mathbf{a}_l(\theta_d)]$ is the full column rank steering matrix with $\mathbf{a}_l(\theta)$ represents the complex manifold of the $l$-th subarray and $\mathbf{\Theta} = [\theta_1 \ \cdots \ \theta_d]^T$ denotes the vector of the unknown DOAs; $\mathbf{x}(t_l) = [x_1(t_l) \ \cdots \ x_d(t_l)]^T$ and $\mathbf{n}_l(t_l)$ are the complex signal amplitudes and additive white sensor noise, respectively; $\mathbf{x}(t_l)$ and $\mathbf{n}_l(t_l)$ are multivariate independent zero-mean wide-sense stationary processes; $\mathbf{n}_l(t_l)$ is assumed spatially white circular complex Gaussian (CG) with $E\{\mathbf{n}_l(t_l)\mathbf{n}_l^H(t_l)\} = \sigma^2\mathbf{I}_{M_l}$, whereas $\mathbf{x}(t_l)$ is NCG with $\mathbf{P} = E\{\mathbf{x}(t_l)\mathbf{x}^H(t_l)\}$ and $\mathbf{Q} = E\{\mathbf{x}(t_l)\mathbf{x}^T(t_l)\}$.

From (1), the two covariance matrices corresponding to the $l$-th subarray are

$$\mathbf{R}_l(\mathbf{\Phi}) = \mathbf{A}_l\mathbf{P}\mathbf{A}_l^H + \sigma^2\mathbf{I}_{M_l} \ \text{ and } \ \mathbf{\Sigma}_l(\mathbf{\Phi}) = \mathbf{A}_l\mathbf{Q}\mathbf{A}_l^T \tag{2}$$

where $\mathbf{\Phi} = [\mathbf{\Theta}^T \ \ \mathbf{\Omega}^T \ \ \sigma^2]^T$ denotes the unknown parameter vector with $\mathbf{\Omega}$ being a real vector made from the free real parameters $\mathbf{P}(i,i)$, $\{\Re(\mathbf{P}(i,j)), \Im(\mathbf{P}(i,j))\}_{j>i}$ and $\{\Re(\mathbf{Q}(i,j)), \Im(\mathbf{Q}(i,j))\}_{j\geq i}$.

Under the assumption that the array is noncoherent across the subarrays, the observed processes between different subarrays are independent, i.e., $E\{\mathbf{y}_l(t_l)\mathbf{y}_k^H(t_k)\} = \delta_{l,k}\mathbf{R}_l$ and $E\{\mathbf{y}_l(t_l)\mathbf{y}_k^T(t_k)\} = \delta_{l,k}\mathbf{\Sigma}_l$. The noncircularity rate $\rho_i$ ($0 \leq \rho_i \leq 1$) of the $i$th source is defined by $E\{x_i^2(t_l)\} = \rho_i e^{j\varphi_{li}} E\{|x_i(t_l)|^2\} = \rho_i e^{j\varphi_{li}}\sigma_i^2$, where $\varphi_{li}$ is the noncircularity phase of the $i$th source at the $l$-th subarray. It is further assumed that the source number $d$ is known and $d < d_{\max}$, where $d_{\max}$ is the greatest number for which the parameter vector $\mathbf{\Phi}$ is identifiable from $\{\mathbf{R}_l, \mathbf{\Sigma}_l\}_{l=1,\cdots,L}$.

### B. ML Estimator

Let $\mathbf{Y} = [\mathbf{Y}_1 \ \cdots \ \mathbf{Y}_L]$ with $\mathbf{Y}_l = [\mathbf{y}_l(1) \ \cdots \ \mathbf{y}_l(N_l)]$. From the above assumptions, the pdf of $\mathbf{Y}$ is [25]

$$p(\mathbf{Y}) = \prod_{l=1}^{L}\prod_{n=1}^{N_l} \frac{1}{\pi^{M_l}\sqrt{\det(\tilde{\mathbf{R}}_l)}} \exp\left(-\frac{1}{2}\tilde{\mathbf{y}}_l^H(n)\tilde{\mathbf{R}}_l^{-1}\tilde{\mathbf{y}}_l(n)\right) \quad (3)$$

where $\tilde{\mathbf{y}}_l(n) = [\mathbf{y}_l^T(n) \ \ \mathbf{y}_l^H(n)]^T$ and

$$\tilde{\mathbf{R}}_l = E\{\tilde{\mathbf{y}}_l(n)\tilde{\mathbf{y}}_l^H(n)\} = \tilde{\mathbf{A}}_l\tilde{\mathbf{P}}\tilde{\mathbf{A}}_l^H + \sigma^2\mathbf{I}_{2M_l} \quad (4)$$

$$\tilde{\mathbf{A}}_l = \begin{bmatrix}\mathbf{A}_l & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_l^*\end{bmatrix} \text{ and } \tilde{\mathbf{P}} = \begin{bmatrix}\mathbf{P} & \mathbf{Q}\\ \mathbf{Q}^* & \mathbf{P}^*\end{bmatrix}. \quad (5)$$

Then, the log-likelihood function can be written as

$$\mathcal{L}(\mathbf{\Phi}) = -\frac{1}{2}\sum_{l=1}^{L} N_l\left(\ln[\det(\tilde{\mathbf{R}}_l)] + \mathrm{Tr}(\tilde{\mathbf{R}}_l^{-1}\hat{\tilde{\mathbf{R}}}_l)\right) \quad (6)$$

where $\hat{\tilde{\mathbf{R}}}_l = 1/N_l\sum_{n=1}^{N_l}\tilde{\mathbf{y}}_l(n)\tilde{\mathbf{y}}_l^H(n)$ is the sample covariance matrix. Unfortunately, we are unable to eliminate the nuisance parameters $\mathbf{\Omega}$ and $\sigma^2$ analytically from the likelihood (6), except for the case of a single source or a single subarray. This motivates us to develop more efficient methods.

### C. Gaussian CRB

The closed-form expression of the CRB for NCG sources is derived in Appendix A as

$$\mathrm{CRB}_{\mathbf{\Theta}}^{\mathrm{NCG}} = 2\left[\tilde{\mathbf{D}}^H\left(\mathbf{C}_{\tilde{\mathbf{r}}}^{-1} - \mathbf{C}_{\tilde{\mathbf{r}}}^{-1}\tilde{\mathcal{A}}(\tilde{\mathcal{A}}^H\mathbf{C}_{\tilde{\mathbf{r}}}^{-1}\tilde{\mathcal{A}})^{-1}\tilde{\mathcal{A}}^H\mathbf{C}_{\tilde{\mathbf{r}}}^{-1}\right)\tilde{\mathbf{D}}\right]^{-1} \quad (7)$$

where $\mathbf{C}_{\tilde{\mathbf{r}}} = \mathrm{diag}\{\mathbf{C}_{\tilde{\mathbf{r}}_1}, \cdots, \mathbf{C}_{\tilde{\mathbf{r}}_L}\}$ with $\mathbf{C}_{\tilde{\mathbf{r}}_l} = 1/N_l\tilde{\mathbf{R}}_l^T \otimes \tilde{\mathbf{R}}_l$, $\tilde{\mathcal{A}} = [\tilde{\mathcal{A}}_1^T \ \cdots \ \tilde{\mathcal{A}}_L^T]^T$ with $\tilde{\mathcal{A}}_l = [(\tilde{\mathbf{A}}_l^* \otimes \tilde{\mathbf{A}}_l)\tilde{\mathbf{U}} \ \ \mathrm{vec}(\mathbf{I}_{2M_l})]$, $\tilde{\mathbf{D}} = [\tilde{\mathbf{D}}_1^T \ \cdots \ \tilde{\mathbf{D}}_L^T]^T$ with the $i$th column of $\tilde{\mathbf{D}}_l$ is given by $\mathrm{vec}\{\tilde{\mathbf{A}}_l\tilde{\mathbf{P}}(\partial\tilde{\mathbf{A}}_l/\partial\theta_i)^H + (\partial\tilde{\mathbf{A}}_l/\partial\theta_i)\tilde{\mathbf{P}}\tilde{\mathbf{A}}_l^H\}$, and $\tilde{\mathbf{U}}$ is a constant column full rank matrix defined as $\mathrm{vec}(\tilde{\mathbf{P}}) = \tilde{\mathbf{U}}\mathbf{\Omega}$. In the particular case of circular signals with $\mathbf{Q} = \mathbf{0}$ and $\mathbf{\Sigma}_l = \mathbf{0}$, the CRB (7) reduces to the CRB for circular signals [11].

## III. Modified GLS Method for Circular Sources

The GLS estimator in [10] is designed for multiple noncoherent subarrays for circular sources. It is asymptotically efficient only for circular Gaussian sources. But for non-Gaussian and/or noncircular sources, it is no longer the optimal SO estimator based on $\{\mathbf{R}_l\}_{l=1,\cdots,L}$. In the section, we modify the GLS estimator [10] to be an AMV SO estimator for circular (Gaussian or non-Gaussian) sources.

The GLS estimator minimizes the total sum of squares of the transformed covariance matching errors as follows [10]

$$\hat{\mathbf{\Phi}} = \arg\min_{\mathbf{\Phi}}\sum_{l=1}^{L}\left\|\sqrt[2]{N_l}\hat{\mathbf{R}}_l^{-\frac{1}{2}}\left(\hat{\mathbf{R}}_l - \mathbf{R}_l\right)\hat{\mathbf{R}}_l^{-\frac{1}{2}}\right\|_F^2 \quad (8)$$

where $\hat{\mathbf{R}}_l$ is the sample estimated version of $\mathbf{R}_l$. Consequently, the GLS method estimates the DOAs as [10]

$$\hat{\mathbf{\Theta}} = \arg\min_{\mathbf{\Theta}}\left\|\left[\mathbf{I} - \mathbf{W}(\bar{\mathbf{W}}^T\bar{\mathbf{W}})^{-1}\Re(\mathbf{W})^T\right]\hat{\mathbf{C}}_{\mathbf{r}}^{-\frac{1}{2}}\hat{\mathbf{r}}\right\|^2 \quad (9)$$

where $\hat{\mathbf{r}} = [\hat{\mathbf{r}}_1^T \ \cdots \ \hat{\mathbf{r}}_L^T]^T$ with $\hat{\mathbf{r}}_l = \mathrm{vec}^T(\hat{\mathbf{R}}_l)$, $\mathbf{W} = \hat{\mathbf{C}}_{\mathbf{r}}^{-1/2}\mathcal{A}$, $\bar{\mathbf{W}} = [\Re^T(\mathbf{W}) \ \ \Im^T(\mathbf{W})]^T$ and $\hat{\mathbf{C}}_{\mathbf{r}} = \mathrm{diag}\{\hat{\mathbf{C}}_{\mathbf{r}_1}, \cdots, \hat{\mathbf{C}}_{\mathbf{r}_L}\}$ with $\hat{\mathbf{C}}_{\mathbf{r}_l} = 1/N_l\hat{\mathbf{R}}_l^T \otimes \hat{\mathbf{R}}_l$.

With the use of $||\mathbf{X}||_F^2 = ||\mathrm{vec}(\mathbf{X})||^2$, $\mathrm{vec}(\mathbf{XYZ}) = (\mathbf{Z}^T \otimes \mathbf{X})\mathrm{vec}(\mathbf{Y})$ and $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$, the GLS method (9) has an alternative expression as

$$\hat{\mathbf{\Phi}} = \arg\min_{\mathbf{\Phi}}[\hat{\mathbf{r}} - \mathbf{r}]^H\hat{\mathbf{C}}_{\mathbf{r}}^{-1}[\hat{\mathbf{r}} - \mathbf{r}]. \quad (10)$$

Since $\mathbf{C}_{\mathbf{r}}$ is the asymptotic covariance of $\hat{\mathbf{r}}$ only for circular Gaussian sources [15], the GLS algorithm is asymptotically efficient only for circular Gaussian sources. Let $\mathbf{G}_{\mathbf{r}_l}$ denote the asymptotic covariance of $\hat{\mathbf{r}}_l$ and from [15], we can modify the GLS algorithm to be an AMV SO algorithm for circular (Gaussian or non-Gaussian) sources by replacing $\hat{\mathbf{C}}_{\mathbf{r}_l}$ in (10) with a consistent estimate of $\mathbf{G}_{\mathbf{r}_l}$, e.g.,

$$\hat{\mathbf{G}}_{\mathbf{r}_l} = \frac{1}{N_l^2}\sum_{n=1}^{N_l}\left[\mathbf{q}_l(n) - \hat{\mathbf{r}}_l\right]\left[\mathbf{q}_l(n) - \hat{\mathbf{r}}_l\right]^H \quad (11)$$

where $\hat{\mathbf{r}}_l = \mathrm{vec}(\hat{\mathbf{R}}_l) = 1/N_l\sum_{n=1}^{N_l}\mathbf{q}_l(n)$ and $\mathbf{q}_l(n) = \mathbf{y}_l^*(n) \otimes \mathbf{y}_l(n)$. Specifically, the modified GLS method has the same expression of (9) with $\hat{\mathbf{C}}_{\mathbf{r}}$ replaced by $\hat{\mathbf{G}}_{\mathbf{r}}$, where $\hat{\mathbf{G}}_{\mathbf{r}} = \mathrm{diag}\{\hat{\mathbf{G}}_{\mathbf{r}_1}, \cdots, \hat{\mathbf{G}}_{\mathbf{r}_L}\}$.

## IV. AMV SO and MUSIC-Like Estimators for Noncircular Sources

In this section, we present an AMV SO estimator and a MUSIC-like estimator for noncircular sources which apply to multiple noncoherent subarrays.

### A. AMV SO Estimator for Noncircular Sources

Since the second covariance matrices $\{\mathbf{\Sigma}_l\}_{l=1,\cdots,L}$ do not vanish for noncircular signals, performance improvement can be obtained by using both the first and second covariance matrices $\{\mathbf{R}_l\}_{l=1,\cdots,L}$ and $\{\mathbf{\Sigma}_l\}_{l=1,\cdots,L}$. Let $\mathbf{s} = [\mathbf{s}_1^T \ \cdots \ \mathbf{s}_L^T]^T$ with $\mathbf{s}_l = [\mathbf{r}_l^T \ \ \mathrm{v}^T(\mathbf{\Sigma}_l) \ \ \mathrm{v}^H(\mathbf{\Sigma}_l)]^T$, we extend the AMV estimator [15] for multiple noncoherent subarrays as

$$\hat{\mathbf{\Phi}} = \arg\min_{\mathbf{\Phi}}[\hat{\mathbf{s}} - \mathbf{s}]^H\hat{\mathbf{G}}_{\mathbf{s}}^{-1}[\hat{\mathbf{s}} - \mathbf{s}] \quad (12)$$

where $\hat{\mathbf{G}}_{\mathbf{s}} = \mathrm{diag}\{\hat{\mathbf{G}}_{\mathbf{s}_1}, \cdots, \hat{\mathbf{G}}_{\mathbf{s}_L}\}$ is a consistent estimate of the covariance of $\hat{\mathbf{s}}$ with $\hat{\mathbf{G}}_{\mathbf{s}_l}$ given by

$$\hat{\mathbf{G}}_{\mathbf{s}_l} = \frac{1}{N_l^2}\sum_{n=1}^{N_l}\left[\tilde{\mathbf{q}}_l(n) - \hat{\mathbf{s}}_l\right]\left[\tilde{\mathbf{q}}_l(n) - \hat{\mathbf{s}}_l\right]^H \quad (13)$$

where $\hat{\mathbf{s}}_l = 1/N_l \sum_{n=1}^{N_l} \tilde{\mathbf{q}}_l(n)$ and $\tilde{\mathbf{q}}_l(n) = [\mathbf{y}_l^H(n) \otimes \mathbf{y}_l^T(n) \quad \mathbf{T}(\mathbf{y}_l^T(n) \otimes \mathbf{y}_l^T(n)) \quad \mathbf{T}(\mathbf{y}_l^H(n) \otimes \mathbf{y}_l^H(n))]^T$. $\mathbf{T}$ is the selection matrix defined as $\mathrm{v}(\cdot) = \mathbf{T}\mathrm{vec}(\cdot)$. Then, similar to the derivation of (9), the AMV estimator (12) can be finally expressed as

$$\hat{\mathbf{\Theta}} = \arg\min_{\mathbf{\Theta}} \left\| \left[ \mathbf{I} - \mathbf{S}(\bar{\mathbf{S}}^T\bar{\mathbf{S}})^{-1}\Re(\mathbf{S})^T \right] \hat{\mathbf{G}}_{\mathbf{s}}^{-\frac{1}{2}}\hat{\mathbf{s}} \right\|^2 \quad (14)$$

where $\hat{\mathbf{s}} = [\hat{\mathbf{s}}_1^T \quad \cdots \quad \hat{\mathbf{s}}_L^T]^T$, $\mathbf{S} = \hat{\mathbf{G}}_{\mathbf{s}}^{-\frac{1}{2}}\breve{\mathcal{A}}$, $\bar{\mathbf{S}} = [\Re^T(\mathbf{S}) \quad \Im^T(\mathbf{S})]^T$, $\breve{\mathcal{A}} = [\breve{\mathcal{A}}_1^T \quad \cdots \quad \breve{\mathcal{A}}_L^T]^T$ where $\breve{\mathcal{A}}_l$ is a known column full rank matrix defined as $\mathbf{s}_l = \breve{\mathcal{A}}_l[\mathbf{\Omega}^T \quad \sigma^2]^T$. For $L = 1$, this method reduces to the AMV method in [15].

### *B. MUSIC-Like Estimator for Noncircular Sources*

When the sources are uncorrelated and strict-sense noncircular (such as BPSK and MASK signals), i.e., $\rho_i = 1$ for $i = 1, \cdots, d$, we have $\mathbf{P} = \mathrm{diag}\{\sigma_1^2, \cdots, \sigma_d^2\}$ and $\mathbf{Q} = \mathbf{\Psi}_l\mathbf{P}$ with $\mathbf{\Psi}_l = \mathrm{diag}\{e^{j\varphi_{1,l}}, \cdots, e^{j\varphi_{d,l}}\}$. $\varphi_{i,l}$ denotes the noncircularity phase of the $i$-th source at the $l$-th subarray. Then, $\tilde{\mathbf{R}}_l$ can be expressed as

$$\tilde{\mathbf{R}}_l = \begin{bmatrix} \mathbf{A}_l \\ \mathbf{A}_l^*\mathbf{\Psi}_l^* \end{bmatrix} \mathbf{P} \left[ \mathbf{A}_l^H \quad \mathbf{\Psi}_l\mathbf{A}_l^T \right] + \sigma^2\mathbf{I}_{2M_l}. \quad (15)$$

Note that, (15) can be alternatively viewed as the case of $d$ uncorrelated circular sources observed from an extended space with an extended steering vector $\tilde{\mathbf{a}}_l(\theta, \varphi_l)$, where $\tilde{\mathbf{a}}_l(\theta, \varphi_l) = [\mathbf{a}_l^T(\theta) \quad \mathbf{a}_l^H(\theta)e^{-j\varphi_l}]^T$. In this scenario, we can extend the w-MUSIC method [11] to get the following NC-MUSIC method.

Specifically, let $\tilde{\mathbf{\Pi}}_l = \sum_{i=d+1}^{2M_l} \tilde{\mathbf{e}}_{l,i}\tilde{\mathbf{e}}_{l,i}^H$ denote the projector onto the noise subspace of $\tilde{\mathbf{R}}_l$ and define $\tilde{\mathbf{U}}_l = \sigma^2 \sum_{i=1}^{d} \tilde{\lambda}_{l,i}/(\sigma^2 - \tilde{\lambda}_{l,i})^2\tilde{\mathbf{e}}_{l,i}\tilde{\mathbf{e}}_{l,i}^H$, where $\tilde{\lambda}_{l,i}$ and $\tilde{\mathbf{e}}_{l,i}$ are the eigenvalues and eigenvectors of $\tilde{\mathbf{R}}_l$ with $\tilde{\lambda}_{l,1} \geq \cdots \geq \tilde{\lambda}_{l,2M_l}$, the NC-MUSIC method estimates the $d$ DOAs via finding the $d$ minima of the function

$$f_2(\theta) = \min_{\varphi_1,\cdots,\varphi_L} \sum_{l=1}^{L} \hat{\tilde{w}}_l(\theta, \varphi_l)\tilde{\mathbf{a}}_l^H(\theta, \varphi_l)\hat{\tilde{\mathbf{\Pi}}}_l\tilde{\mathbf{a}}_l(\theta, \varphi_l) \quad (16)$$

where $\hat{\tilde{w}}_l(\theta, \varphi_l) = N_l[\tilde{\mathbf{a}}_l^H(\theta, \varphi_l)\hat{\tilde{\mathbf{U}}}_l\tilde{\mathbf{a}}_l(\theta, \varphi_l)]^{-1}$ with $\hat{\tilde{\mathbf{\Pi}}}_l$ and $\hat{\tilde{\mathbf{U}}}_l$ be respectively the estimates of $\tilde{\mathbf{\Pi}}_l$ and $\tilde{\mathbf{U}}_l$ from the sample covariance $\hat{\tilde{\mathbf{R}}}_l$, i.e., made of $\hat{\tilde{\lambda}}_{l,i}$, $\hat{\tilde{\mathbf{e}}}_{l,i}$ and $\hat{\sigma}^2 = \left[\sum_{l=1}^{L}(2M_l - d)\right]^{-1} \sum_{l=1}^{L}\sum_{i=d+1}^{2M_l} \hat{\tilde{\lambda}}_{l,i}$. NC-MUSIC combines the MUSIC spectrum of the subarrays with weights $\hat{\tilde{w}}_l$, $l = 1, \cdots, L$. This formulation can be derived from the ML criterion as [11].

To implement the NC-MUSIC method efficiently, $e^{-j\varphi_l}$ in (16) can be replaced by its estimate $e^{-j\hat{\varphi}_l} = -\mathbf{a}_l^T(\theta)\hat{\mathbf{\Pi}}_{l,2}^*\mathbf{a}_l(\theta)[\mathbf{a}_l^H(\theta)\hat{\mathbf{\Pi}}_{l,1}\mathbf{a}_l(\theta)]^{-1}$ [14], where $\hat{\mathbf{\Pi}}_{l,1}$ and $\hat{\mathbf{\Pi}}_{l,2}$ are partitioned from $\hat{\tilde{\mathbf{\Pi}}}_l$ as $\hat{\tilde{\mathbf{\Pi}}}_l = \begin{bmatrix} \hat{\mathbf{\Pi}}_{l,1} & \hat{\mathbf{\Pi}}_{l,2} \\ \hat{\mathbf{\Pi}}_{l,2}^* & \hat{\mathbf{\Pi}}_{l,1}^* \end{bmatrix}$. Then, the NC-MUSIC method estimates the $d$ DOAs by finding the $d$ minima of the function

$$f_2(\theta) = \sum_{l=1}^{L} \hat{\tilde{w}}_l(\theta, \hat{\varphi}_l)\tilde{\mathbf{a}}_l^H(\theta, \hat{\varphi}_l)\hat{\tilde{\mathbf{\Pi}}}_l\tilde{\mathbf{a}}_l(\theta, \hat{\varphi}_l). \quad (17)$$

For $L = 1$, this method reduces to the MUSIC method in [16].

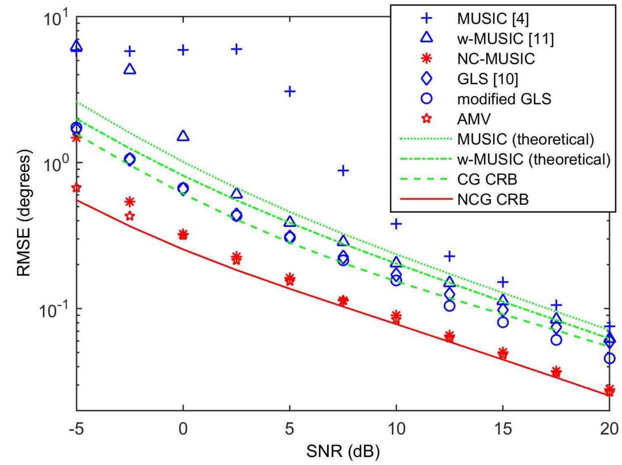


Fig. 1. RMSE and theoretical performance for different SNR with two uncorrelated BPSK sources ($\rho = 1$).

## V. Simulation Results

We consider $L = 4$ noncoherent subarrays with each subarray being a uniform linear array (ULA) and $M_1 = \cdots = M_4 = 6$. The interelement spacing of the subarrays are 0.3, 0.35, 0.45 and 0.5 wavelengths, respectively. Two equally powered sources having identical noncircularity rate (i.e., $\rho_1 = \rho_2 = \rho$) with noncircularity phases $\varphi_{li}$ randomly selected from $[0 \quad 2\pi]$ are considered. The sample covariance matrices $\hat{\mathbf{R}}_l \in \mathbb{C}^{6\times6}$, $\hat{\mathbf{\Sigma}}_l \in \mathbb{C}^{6\times6}$ and $\hat{\tilde{\mathbf{R}}}_l \in \mathbb{C}^{12\times12}$ are calculated for each local subarray. The GLS [10], MUSIC [4], w-MUSIC [11], and modified GLS methods are based on $\{\hat{\mathbf{R}}_l\}_{l=1,\cdots,4}$, the AMV estimator (14) is based on $\{\hat{\mathbf{R}}_l, \hat{\mathbf{\Sigma}}_l\}_{l=1,\cdots,4}$, while the NC-MUSIC (17) is based on $\{\hat{\tilde{\mathbf{R}}}_l\}_{l=1,\cdots,4}$. Each result is an average over 500 independent runs. When the two sources are *a priori* known to be uncorrelated, the parameter model

$$\mathbf{\Omega} = [\{\mathbf{P}(i,i), \Re(\mathbf{Q}(i,i)), \Im(\mathbf{Q}(i,i))\}_{i=1,\cdots,d}]^T \quad (18)$$

is used. For correlated sources, $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega} = [\{\Re(\mathbf{P}(i,j)), \Im(\mathbf{P}(i,j)), \Re(\mathbf{Q}(i,j)), \Im(\mathbf{Q}(i,j))\}_{1\leq j<i<d}, \{\mathbf{P}(i,i), \Re(\mathbf{Q}(i,i)), \Im(\mathbf{Q}(i,i))\}_{i=1,\cdots,d}]^T. \quad (19)$$

In the first experiment, two uncorrelated sources are considered. The DOA separation between the two sources, denoted by $\Delta\theta = \theta_2 - \theta_1$, is 0.1 rad with $\theta_1 = -0.05$ rad and $\theta_2 = 0.05$ rad relative to boresight. The numbers of snapshots taken at the subarrays are 200, 400, 800 and 1200, respectively.

Fig. 1 compares the experimental and theoretical performance of the algorithms for two uncorrelated BPSK sources ($\rho = 1$). The AMV method (14) evidently outperforms the GLS and modified GLS methods and gives the best performance among the compared methods. Although NC-MUSIC has comparable performance with the AMV method at high SNR, it shows distinctly worse threshold performance than the AMV method. Moreover, the modified GLS outperforms the GLS [10] at relatively high SNR (e.g., SNR $>$ 10 dB), which can be explained as follows. Based on $\{\mathbf{R}_l\}_{l=1,\cdots,L}$ only, the GLS [10] is optimal only for CG sources, however, the Gaussian assumption can

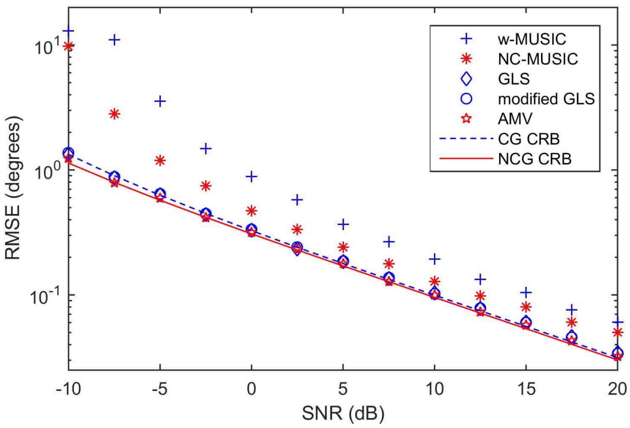


Fig. 2. RMSE versus SNR with two correlated BPSK sources with correlation coefficient of 0.8.

yield important performance losses at high SNR for phase-shift keying (PSK) or continuous-phase modulation (CPM) signals [26].

In the second experiment, two correlated BPSK sources with correlation coefficient of 0.8 and $\Delta\theta = 0.2$ rad are considered. Accordingly, the parameter model (19) is used. Fig. 2 shows the RMSE of w-MUSIC, NC-MUSIC, GLS, modified GLS and AMV versus SNR. It can be seen that, for correlated sources, the advantage of the proposed noncircular methods (NC-MUSIC and AMV) over the circular methods diminishes, while the GLS and modified GLS algorithms generally give comparable performance.

## VI. Conclusion

The modified GLS estimator outperforms the traditional GLS estimator only for uncorrelated sources and at high SNR conditions. Compared to the GLS method, the AMV estimator shows evident performance advantage mainly for uncorrelated sources. The NC-MUSIC estimator significantly outperforms the w-MUSIC estimator for strict-sense noncircular sources.

## Appendix

From the noncircular complex Gaussian Slepian-Bangs formula [17], the Fisher information matrix is

$$\mathrm{FIM}_{i,j} = \frac{1}{2}\sum_{l=1}^{L} N_l \mathrm{Tr}\left[\left(\partial\tilde{\mathbf{R}}_l/\partial\Phi_i\right)\tilde{\mathbf{R}}_l^{-1}\left(\partial\tilde{\mathbf{R}}_l/\partial\Phi_j\right)\tilde{\mathbf{R}}_l^{-1}\right].$$

Let $\tilde{\mathbf{r}} = [\tilde{\mathbf{r}}_1^T \quad \cdots \quad \tilde{\mathbf{r}}_L^T]^T$ with $\tilde{\mathbf{r}}_l = \mathrm{vec}^T(\tilde{\mathbf{R}}_l)$, it follows that

$$\mathrm{FIM} = 1/2\left(\partial\tilde{\mathbf{r}}/\partial\mathbf{\Phi}^T\right)^H \mathbf{C}_{\tilde{\mathbf{r}}}^{-1}\left(\partial\tilde{\mathbf{r}}/\partial\mathbf{\Phi}^T\right).$$

Then, with $\tilde{\mathbf{r}}_l = [\tilde{\mathbf{A}}_l^* \otimes \tilde{\mathbf{A}}_l]\mathrm{vec}(\tilde{\mathbf{P}}) + \sigma^2\mathrm{vec}(\mathbf{I}_{2M_l})$, we have $\partial\tilde{\mathbf{r}}/\partial\mathbf{\Phi}^T = [\tilde{\mathbf{D}} \quad \tilde{\mathcal{A}}]$. Consequently, the $\mathbf{\Theta}$-block of $\mathrm{FIM}^{-1}$ is given by (7) using the partitioned matrix inversion lemma.

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