Improved Subspace Direction-of-Arrival Estimation in Unknown Nonuniform Noise Fields

Fei Wen, Umer Javed, Yuan Yang, Di He and Yi Zhang

***Abstract*—This paper improves the classic subspace DOA estimation methods to combat unknown nonuniform noise. Utilizing an approximate orthogonality between the signal subspace and a tailored eigen-space of the array covariance matrix in high signal-to-noise ratio (SNR) conditions, we modify the classic multiple signal classification (MUSIC) and root-MUSIC algorithms to be competent in unknown nonuniform noise. Compared to the MUSIC and root-MUSIC methods, the proposed methods are able to achieve significant better performance in unknown nonuniform noise environments. Simulation results show that the two proposed methods significantly outperform the MUSIC and root-MUSIC methods in the whole SNR range and approach the Cramer-Rao bound (CRB) at high SNR.**

***Index Terms*—Array signal processing, direction-of-arrival estimation, minimum variance, multiple signal classification, nonuniform noise**

I. Introduction

Estimating the direction-of-arrival (DOA) of multiple sources using an array is an essential task in many applications, such as radar, sonar, communications, geophysics, tracking and localization [1], [2], [18]–[21]. Among the traditional algorithms, the subspace algorithms, such as the well-known multiple signal classification (MUSIC) [3] and ESPRIT [4], which are computationally efficient and have better resolution than the conventional beamformers, are of the most popular.

A common assumption used in traditional subspace DOA estimation techniques, such as the well-known MUSIC and ESPRIT, is the so-called uniform white noise assumption. This assumption leads to performance lose of these methods in the applications where the noise is not spatially uniform but has an arbitrary diagonal covariance matrix. Nonuniform noise may be caused by the variation of the manufacturing process or the imperfection of array calibration. In such applications, if the noise is perfectly known, the noise model can be easily transformed to be uniform. Then, the classic subspace algorithms can be straightforwardly applied by taking this transformation step into account [5]. On the other hand, if the noise is unknown, the noise variances may be estimated by collecting signal-free data from the sensor array. However, in practical applications, the noise parameters may be time-varying, and a signal-free environment is not always available.

There are mainly two classes of direction finding techniques exist for unknown nonuniform noise, including the covariance differencing [6]–[9] and the maximum likelihood (ML) techniques [10]–[12]. The covariance differencing idea is firstly introduced in [6], which is not practical for some applications since it usually requires some sort of translation or rotation of the array to obtain the second measurement of the covariance matrix. Then, improved versions are given by [7], [8] which require no multiple measurements of the covariance matrix. But it is restricted to uniformly correlated noise since the noise covariance matrix is assumed as a symmetric Toeplitz matrix. Moreover, a transform-based version [9] is proposed to completely remove spatially nonuniform noise, but its performance depends on the parameter choice for the translation matrix.

The ML methods estimate the parameters of noise and signals simultaneously by optimizing various likelihood cost functions and have excellent asymptotic and threshold performance [10]–[12]. However, the ML algorithms usually require the processing of high-dimensional nonlinear optimization, which leads to a heavy computational burden. Besides the above methods that apply to nonuniform noise, there are other methods designed to cope with unknown colored noise [13]. Moreover, the covariance differencing approach has been extended to wideband signals for unknown colored noise [14].

The goal of this work is to develop computationally simple DOA estimators for unknown nonuniform noise environments. First, we show that the minimum variance distortionless response (MVDR) spectrum reduces to the MUSIC spectrum of the perfectly prewhitened array outputs in high signal-to-noise ratio (SNR) conditions. Then, an approximate orthogonality between the signal subspace and a tailored eigen-space of the array covariance matrix for high SNRs is obtained. Based on this approximate orthogonality, the MUSIC and root-MUSIC methods are modified to gain significant improvement in nonuniform noise environments.

II. Standard subspace estimators

Consider an M sensor arbitrarily distributed array receiving d $(d < M)$ narrowband far-field signals impinging with unknown DOAs $\theta_1, \theta_2, \cdots, \theta_d$. The $M \times 1$ complex snapshot vector of the tth array outputs can be modeled as

$$\mathbf{x}(t) = \mathbf{A}(\theta)\mathbf{s}(t) + \mathbf{n}(t), \qquad t = 1, \cdots, N \tag{1}$$

This work is supported by the Important National Science and Technology Specific Project of China under Grant No. 2016ZX03001022-006, the National Natural Science Foundation of China under grants 61401501, 61472442, 61304225, 61573242, 91438113, and 61402283, the National Science and Technology Major Project under grants GFZX0301010708, and the Shanghai Science and Technology Committee under grants 16DZ1100402 and 15511105100.

F. Wen, U. Javed and D. He are with the Department of Electronic Engineering, Shanghai Jiao Tong University, Shanghai 200240, China (e-mail: wenfei, umerjaved, dihe@sjtu.edu.cn).

F. Wen and Y. Yang are with the Air Force Engineering University, Xian 710000, China.

Y. Zhang is with Huawei Technologies Co. Ltd., Shanghai, P.R. China (e-mail: aaabear@huawei.com).

where $\mathbf{A}(\theta) = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_d)]$ is the $M \times d$ full column rank steering matrix with $\mathbf{a}(\theta)$ represents the array's $M \times 1$ complex manifold and $\theta = [\theta_1, \cdots, \theta_d]^T$ denotes the $d \times 1$ vector of the unknown DOAs, $\mathbf{s}(t)$ is the $d \times 1$ vector of complex signal amplitudes, $\mathbf{n}(t)$ is the $M \times 1$ vector of additive white sensor noise, $(\cdot)^T$ stands for the transpose, and N is the number of snapshots. The sensor noise is assumed to be a zero-mean temporally white Gaussian process with the unknown diagonal covariance matrix

$$\mathbf{Q} = E\{\mathbf{n}(t)\mathbf{n}^H(t)\} = \mathrm{diag}\{\sigma_1^2, \sigma_2^2, \cdots, \sigma_M^2\}. \quad (2)$$

$(\cdot)^H$ stands for the Hermitian transpose.

Assuming the signals and noise are uncorrelated, the array covariance matrix is

$$\mathbf{R} = E\{\mathbf{x}(t)\mathbf{x}^H(t)\} = \mathbf{APA}^H + \mathbf{Q} \quad (3)$$

with $\mathbf{P} = E\{\mathbf{s}(t)\mathbf{s}^H(t)\}$ is the covariance matrix of the emitter signals, which is assumed to be full rank d (no coherent signals). Let $\mathbf{EVE}^H$ be the eigendecomposition of $\mathbf{R}$, where $\mathbf{V} = \mathrm{diag}\{\lambda_1, \cdots, \lambda_M\}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M$ be the engivalues of $\mathbf{R}$, and $\mathbf{E} = [\mathbf{e}_1, \cdots, \mathbf{e}_M]$ with $\mathbf{e}_i$ be the normalized eigenvector corresponding to λ_i. The eigenvectors are assumed to form an orthonormal basis, i.e., $\mathbf{EE}^H = \mathbf{E}^H\mathbf{E} = \mathbf{I}$.

If the noise is spatially white, that is each sensor has uniform noise variance σ^2 with $\mathbf{Q} = \sigma^2\mathbf{I}$, the eigendecomposition of $\mathbf{R}$ has the following form

$$\mathbf{R} = \mathbf{EVE}^H = \mathbf{E}_s\mathbf{V}_s\mathbf{E}_s^H + \sigma^2\mathbf{E}_n\mathbf{E}_n^H \quad (4)$$

where $\mathbf{V}_s = \mathrm{diag}\{\lambda_1, \cdots, \lambda_d\}$, $\lambda_1 \geq \cdots \geq \lambda_{d+1} = \cdots = \lambda_M = \sigma^2$, $\mathbf{E}_s = [\mathbf{e}_1, \cdots, \mathbf{e}_d]$ and $\mathbf{E}_n = [\mathbf{e}_{d+1}, \cdots, \mathbf{e}_M]$. By exploiting the property that the signal subspace is orthogonal with the noise subspace, i.e., $\mathrm{span}\{\mathbf{E}_s\} = \mathrm{span}\{\mathbf{A}\} \perp \mathrm{span}\{\mathbf{E}_n\}$, a class of high-resolution spatial spectrum estimators have been proposed. For example, the MUSIC spectrum is defined as [3]

$$f_{\mathrm{MU}}(\theta) = \{\mathbf{a}^H(\theta)\mathbf{E}_n\mathbf{E}_n^H\mathbf{a}(\theta)\}^{-1}. \quad (5)$$

The unknown DOAs can be estimated by a one dimensional maximum search of the spectrum (5).

The MUSIC estimator has been proven to be a large sample realization of the ML estimation which achieves the CRB for uncorrelated signals and sufficiently large N [15]. However, when the noise is not spatially white, as modeled in (2), the orthogonality assumption does not hold again. In this situation, the conventional subspace algorithms can not be expected to give good performance. However, if $\mathbf{Q}$ is known, the array outputs can be prewhitened as

$$\widetilde{\mathbf{x}}(t) = \mathbf{Q}^{-1/2}\mathbf{x}(t) = \mathbf{Q}^{-1/2}\mathbf{A}(\theta)\mathbf{s}(t) + \mathbf{Q}^{-1/2}\mathbf{n}(t). \quad (6)$$

Then, the covariance matrix of the transformed data has the eigendecomposition as

$$\begin{aligned}\widetilde{\mathbf{R}} &= \mathbf{Q}^{-1/2}\mathbf{APA}^H\mathbf{Q}^{-1/2} + \mathbf{I} \\ &= \widetilde{\mathbf{E}}\widetilde{\mathbf{V}}\widetilde{\mathbf{E}}^H = \widetilde{\mathbf{E}}_s\widetilde{\mathbf{V}}_s\widetilde{\mathbf{E}}_s^H + \widetilde{\mathbf{E}}_n\widetilde{\mathbf{E}}_n^H\end{aligned} \quad (7)$$

where $\widetilde{\mathbf{V}} = \mathrm{diag}\{\widetilde{\lambda}_1, \cdots, \widetilde{\lambda}_M\}$, $\widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_d > \widetilde{\lambda}_{d+1} = \cdots = \widetilde{\lambda}_M = 1$, $\widetilde{\mathbf{V}}_s = \mathrm{diag}\{\widetilde{\lambda}_1, \cdots, \widetilde{\lambda}_d\}$, and $\widetilde{\mathbf{E}} = [\widetilde{\mathbf{E}}_s \; \widetilde{\mathbf{E}}_n]$ with $\widetilde{\mathbf{E}}_s = [\widetilde{\mathbf{e}}_1, \cdots, \widetilde{\mathbf{e}}_d]$ and $\widetilde{\mathbf{E}}_n = [\widetilde{\mathbf{e}}_{d+1}, \cdots, \widetilde{\mathbf{e}}_M]$. In this case, the subspace relationships are $\mathrm{span}\{\widetilde{\mathbf{E}}_s\} = \mathrm{span}\{\mathbf{Q}^{-1/2}\mathbf{A}\} \perp \mathrm{span}\{\widetilde{\mathbf{E}}_n\}$, and the MUSIC spectrum becomes

$$\widetilde{f}_{\mathrm{MU}}(\theta) = \{\mathbf{a}^H(\theta)\mathbf{Q}^{-1/2}\widetilde{\mathbf{E}}_n\widetilde{\mathbf{E}}_n^H\mathbf{Q}^{-1/2}\mathbf{a}(\theta)\}^{-1}. \quad (8)$$

In practical applications, the noise parameters may be time-varying and a signal-free environment to estimate the noise covariance matrix $\mathbf{Q}$ is not always available. In section 4, we will propose subspace methods which do not require the information of $\mathbf{Q}$ and avoid the prewhitening step (6).

III. Minimum variance estimator

The MVDR approach is derived by finding the beamformer weight vector to pass a plane wave in the incidence direction with unity gain while minimizing the beam power. The MVDR spectrum is given by

$$f_{\mathrm{MV}}(\theta) = \{\mathbf{a}^H(\theta)\mathbf{R}^{-1}\mathbf{a}(\theta)\}^{-1}. \quad (9)$$

which, in the case of uniform noise and with the use of (4), has an alternative subspace expression

$$f_{\mathrm{MV}}(\theta) = \sigma^2\{\mathbf{a}^H(\theta)(\sigma^2\mathbf{E}_s\mathbf{V}_s^{-1}\mathbf{E}_s^H + \mathbf{E}_n\mathbf{E}_n^H)\mathbf{a}(\theta)\}^{-1}. \quad (10)$$

It is easy to see that the MVDR spectrum approximately reduce to the MUSIC spectrum (5) at high SNR conditions since $\sigma^2/\lambda_i \to 0$ for $i = 1, \cdots, d$.

When the noise is not spatially white, the second equality in (4) and the equality in (10) do not hold again. In this case, with the use of (7), the MVDR spectrum (9) can be expressed as

$$\begin{aligned}f_{\mathrm{MV}}(\theta) &= \{\mathbf{a}^H(\theta)\mathbf{Q}^{-1/2}\widetilde{\mathbf{E}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{E}}^H\mathbf{Q}^{-1/2}\mathbf{a}(\theta)\}^{-1} \\ &= \{\mathbf{a}^H(\theta)\mathbf{Q}^{-1/2}(\widetilde{\mathbf{E}}_s\widetilde{\mathbf{V}}_s^{-1}\widetilde{\mathbf{E}}_s^H + \widetilde{\mathbf{E}}_n\widetilde{\mathbf{E}}_n^H)\mathbf{Q}^{-1/2}\mathbf{a}(\theta)\}^{-1}\end{aligned} \quad (11)$$

since $\mathbf{R}^{-1} = \mathbf{Q}^{-1/2}\widetilde{\mathbf{R}}^{-1}\mathbf{Q}^{-1/2} = \mathbf{Q}^{-1/2}\widetilde{\mathbf{E}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{E}}^H\mathbf{Q}^{-1/2}$. It can be observed that the MVDR spectrum (11) approximately reduce to the MUSIC spectrum (8) at high SNRs, and thus can be expected to give good performance as its MUSIC counterpart (8) at high SNRs in nonuniform noise conditions. However, unlike the subspace approaches requiring the prewhitening step (6), the MVDR approach has a considerable advantage of avoiding such a process and does not require any prior information about the actual noise variances.

IV. Improved subspace estimators

In this section, following the preceding analysis, an approximate orthogonality between the signal subspace and a tailored eigen-space of $\mathbf{R}$ is established for high SNRs under the nonuniform noise model. Then, the conventional MUSIC and root-MUSIC algorithms are improved to be capable of giving good performance in nonuniform noise while avoiding the prewhitening process (6).

From the eigendecomposition of $\mathbf{R}$ and $\widetilde{\mathbf{R}}$, and with the use of $\mathbf{R}^{-1} = \mathbf{Q}^{-1/2}\widetilde{\mathbf{R}}^{-1}\mathbf{Q}^{-1/2}$, we have

$$\begin{aligned}\mathbf{E}\mathbf{V}^{-1}\mathbf{E}^H &= \mathbf{Q}^{-1/2}\widetilde{\mathbf{E}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{E}}^H\mathbf{Q}^{-1/2}\\ &= \mathbf{E}_s\mathbf{V}_s^{-1}\mathbf{E}_s^H + \mathbf{E}_n\mathbf{V}_n^{-1}\mathbf{E}_n^H \\ &= \mathbf{Q}^{-1/2}\widetilde{\mathbf{E}}_s\widetilde{\mathbf{V}}_s^{-1}\widetilde{\mathbf{E}}_s^H\mathbf{Q}^{-1/2} + \mathbf{Q}^{-1/2}\widetilde{\mathbf{E}}_n\widetilde{\mathbf{E}}_n^H\mathbf{Q}^{-1/2}\end{aligned} \tag{12}$$

where $\mathbf{V}_n = \text{diag}\{\lambda_{d+1}, \cdots, \lambda_M\}$ contains the $M-d$ smallest eigenvalues of $\mathbf{R}$. It is worthy to mention that here $\mathbf{E}_s$ (and $\mathbf{E}_n$, respectively) in general has components in both the true signal and noise subspaces due to the nonuniform noise. Under the assumption that the SNR is sufficiently high, we have $\mathbf{E}_s\mathbf{V}_s^{-1}\mathbf{E}_s^H \approx \mathbf{0}$ and $\widetilde{\mathbf{E}}_s\widetilde{\mathbf{V}}_s^{-1}\widetilde{\mathbf{E}}_s^H \approx \mathbf{0}$ since $\lambda_i^{-1} \approx 0$ and $\widetilde{\lambda}_i^{-1} \approx 0$ for $i = 1, \cdots, d$, then the following approximations are obtained

$$\mathbf{E}_n\mathbf{V}_n^{-1}\mathbf{E}_n^H \approx \mathbf{Q}^{-1/2}\widetilde{\mathbf{E}}_n\widetilde{\mathbf{E}}_n^H\mathbf{Q}^{-1/2} \tag{13}$$

and

$$\mathbf{A}^H\mathbf{E}_n\mathbf{V}_n^{-1}\mathbf{E}_n^H\mathbf{A} \approx \mathbf{A}^H\mathbf{Q}^{-1/2}\widetilde{\mathbf{E}}_n\widetilde{\mathbf{E}}_n^H\mathbf{Q}^{-1/2}\mathbf{A} = 0 \tag{14}$$

since $\text{span}\{\mathbf{Q}^{-1/2}\mathbf{A}\} \perp \text{span}\{\widetilde{\mathbf{E}}_n\}$. Equation (14) implies that $\text{span}\{\mathbf{A}\}$ is approximately orthogonal with $\mathbf{E}_n\mathbf{V}_n^{-1/2}$, which is the basis of the proposed subspace methods.

Although the approximation is proposed for high SNR conditions, it will be shown by simulations later that the introduced methods also perform adequately well in relatively low SNRs. With the use of (14), we propose the following modified MUSIC spectrum

$$f'_{\text{MU}}(\theta) = \{\mathbf{a}^H(\theta)\mathbf{E}_n\mathbf{V}_n^{-1}\mathbf{E}_n^H\mathbf{a}(\theta)\}^{-1}. \tag{15}$$

For the particular case of a uniform linear array (ULA), the root-MUSIC [16] method, a variation of MUSIC, is prevalent due to its computational efficiency. The array response of a ULA can be written as

$$\mathbf{a}(\theta) = \mathbf{a}_p(z) = [1\ z^{-1}\ \cdots\ z^{-(M-1)}]^T \tag{16}$$

where $z = e^{j2\pi\Delta\sin(\theta)}$ with Δ be the array element separation in wavelengths. Then, the term $\mathbf{a}^H(\theta)\mathbf{E}_n\mathbf{V}_n^{-1}\mathbf{E}_n^H\mathbf{a}(\theta)$ in (16) can be written as a polynomial in z of order $2M-2$ as

$$P_{\text{root-MU}}(z) = \mathbf{a}^T(z^{-1})\mathbf{E}_n\mathbf{V}_n^{-1}\mathbf{E}_n^H\mathbf{a}_p(z). \tag{17}$$

since $z^H = z^{-1}$. Then, among the $2M-2$ roots of the polynomial (17), the d roots with modulus nearest to unity are selected to be the estimates of the values of z corresponding to the true DOAs.

It is easy to see that, in uniform noise, the proposed MUSIC (15) and root-MUSIC (17) estimators degenerate to the conventional MUSIC and root-MUSIC estimators, respectively, since $\mathbf{V}_n$ (contains the $M-d$ smallest eigenvalues of $\mathbf{R}$) is a scaled identity matrix in this case, i.e., $\mathbf{V}_n = \sigma^2\mathbf{I}$. However, in nonuniform noise, the two proposed methods employ $\mathbf{V}_n$ (has non-identical diagonal elements) to weight the noise subspace, which improves their performance since as indicated by (14) that $\text{span}\{\mathbf{A}\}$ is approximately orthogonal with $\mathbf{E}_n\mathbf{V}_n^{-1/2}$ in this case. In practice, $\mathbf{E}_n$ and $\mathbf{V}_n$ are obtained form the engendecomposition of the sample covariance matrix

$$\hat{\mathbf{R}} = \frac{1}{N}\sum_{t=1}^{N}\mathbf{x}(t)\mathbf{x}^H(t). \tag{18}$$

V. Simulations

A ULA of eight omni-directional sensors with interelement spacing of a half-wavelength is considered. The true DOAs of two equally powered narrowband sources are $\theta_1 = -7^\circ$ and $\theta_2 = 7^\circ$ relative to the array broadside. Mutually independent zero-mean white Gaussian noise is added to each channel to adjust the noise variances to be

$$\mathbf{Q} = \sigma^2\text{diag}\{1, 10, 1, 4, 7, 8, 1, 16\}. \tag{19}$$

The SNR of the first sensor is denoted by $\text{SNR1} = \sigma_s^2/\sigma^2$, where σ_s^2 is the signal variance. The performance is investigated in terms of the RMSE over 500 Monte Carlo runs. The MVDR, MUSIC and proposed MUSIC approaches are implemented by a coarse search of 1° step size in the range of $(-90^\circ \sim 90^\circ)$ followed by a fine search around the angle estimated in the coarse search.

Fig. 1 plots ten typical spectrum estimates of the MUSIC, proposed MUSIC and MVDR approaches with $N = 2000$ snapshots. In the first condition of SNR1 = 0 dB with $\theta_1 = -7^\circ$ and $\theta_2 = 7^\circ$, the MUSIC algorithm fails to resolve the two DOAs. However, the MVDR and proposed MUSIC spectra always have two distinct peaks, which demonstrates their better resolution than MUSIC. In the second condition with a higher SNR, SNR1 = 10 dB, the two DOAs are set to be closer with $\theta_1 = -3.5^\circ$ and $\theta_2 = 3.5^\circ$. In this case, only the proposed MUSIC method is able to distinguish the two sources, which demonstrates its better resolution than the MUSIC and MVDR methods.

Fig. 2 shows the RMSE of the compared methods for varying SNR1. The analytical performance of MVDR [17] and the CRB are also plotted. It is obvious that the two proposed methods have apparent lower RMSE than their MUSIC and root-MUSIC counterparts in the whole SNR range. Meanwhile, the RMSE of the two proposed methods approach the CRB at relatively high SNRs (e.g., SNR1 > 10 dB). Moreover, similar to the relationship between root-MUSIC and MUSIC, the improved root-MUSIC method performs better than the improved MUSIC method at relatively low SNRs. This is reasonable since the RMSE threshold of root-MUSIC, which is due to the choosing of a spurious root of the polynomial, appears at much lower SNR than the threshold of MUSIC, which is due to the loss-of-resolution effect.

The effect of the data size N is examined in Fig. 3 for SNR1 = 20 dB. We can observe that the two proposed estimators, together with the MVDR estimator, obviously perform better than the MUSIC and root-MUSIC estimators when $N > 100$. Furthermore, their RMSE closely approach the CRB when the number of snapshots is sufficient large (i.e., $N > 1000$).

VI. Conclusion

In the case of unknown nonuniform noise, the traditional subspace methods (e.g., MUSIC and root-MUSIC) become

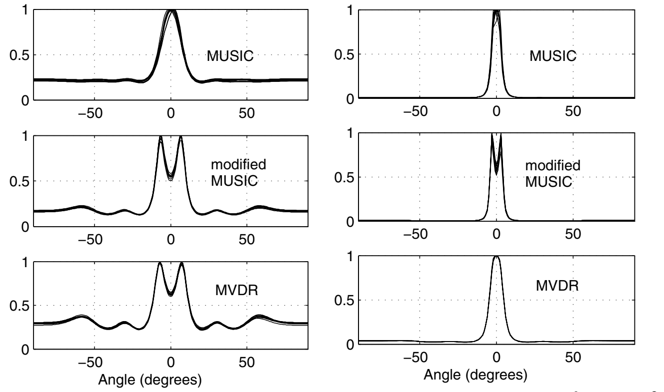

Fig. 1. Ten typical spectrum estimates in nonuniform noise, (a) $\theta_1 = -7°$, $\theta_2 = 7°$ and SNR1 = 0 dB, (b) $\theta_1 = -3.5°$, $\theta_2 = 3.5°$ and SNR1 = 10 dB.

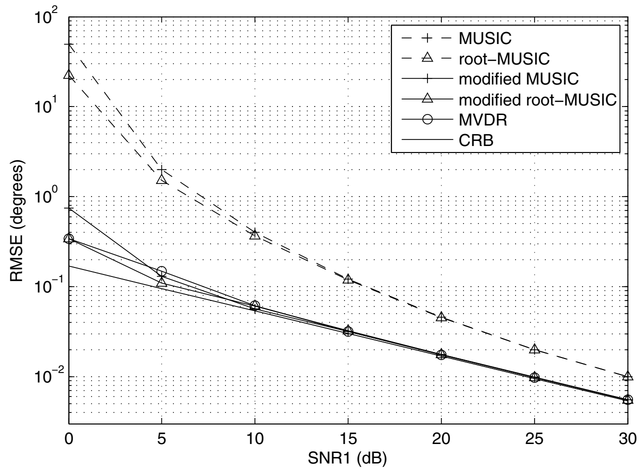

Fig. 2. RMSE versus SNR1 in nonuniform noise with $N = 2000$.

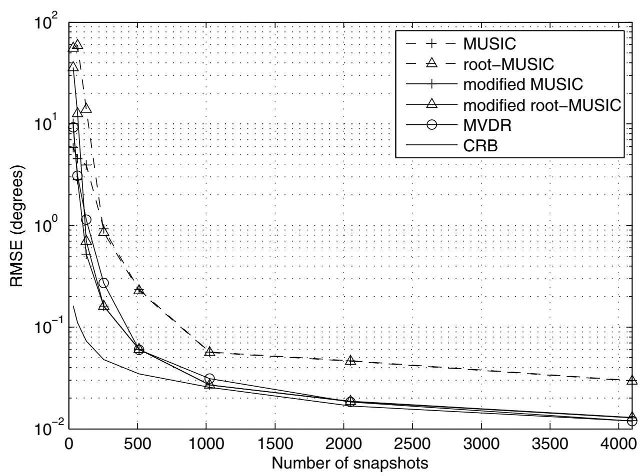

Fig. 3. RMSE for varying N in nonuniform noise with SNR1 = 20 dB.

poor. In this paper, we show that the MUSIC and root-MUSIC methods can be modified to gain significant improvement in nonuniform noise fields. This kind of modification is also applicable to other subspace algorithms. More importantly, the modified MUSIC and root-MUSIC do not require any *a prior* information about the actual variances of the nonuniform noise. The proposed modification fundamentally enhances the applicability of the subspace DOA estimation methods in practical applications.

References

[1] H. Krim, M. Viberg, "Two decades of array signal processing research: The parametric approach," *IEEE Signal Process. Mag.*, vol. 13, no. 4, pp. 67–94, 1996.

[2] F. Wen, Q. Wan, R. Fan, H. Wei, "Improved MUSIC algorithm for multiple noncoherent subarrays," *IEEE Signal Processing Letters*, vol. 21, no. 5, pp. 527–530, 2014.

[3] R.O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas Propagat.*, vol. AP-34, pp. 276–280, Mar. 1986.

[4] R. Roy, A. Paulraj, T. Kailath, "ESPRIT-A subspace rotation approach to estimation of parameters of cisoids in noise," *IEEE Trans. Acoust Speech Signal Processing*, ASSP-34(4), pp. 1340–1342, 1986.

[5] B. Friedlander and A. J. Weiss, "Direction finding using noise covariance modeling," *IEEE Trans. Signal Processing*, vol. 43, pp. 1557–1567, July 1995.

[6] A. Paulraj and T. Kailath, "Eigenstructure method for direction of arrival estimation in the presence of unknown noise fields," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-34, no. 1, pp. 13–20, Jan. 1986.

[7] S. Prasad and R.T. Williams, "A transform-based covariance differencing approach for some classes of parameter estimation problems," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 36, no. 5, pp. 631–641, May 1988.

[8] R. Rajagopal and P.R. Rao, "Generalised algorithm for DOA estimation in a passive sonar," *IEE Proceedings-F*, vol. 140, no. 1, pp. 12–20, Jan. 1993.

[9] A. Moghaddamjoo, "Transform-based covariance differencing approach to the array with spatially nonstationary noise," *IEEE Transactions on Signal Processing*, vol. 39, no. 1, pp. 219–221, Jan. 1991.

[10] M. Pesavento and A.B. Gershman, "Maximum-likelihood direction-of-arrival estimation in the presence of unknown nonuniform noise," *IEEE Trans. on Signal Processing*, vol. 49, no. 7, pp. 1310–1324, Jul. 2001.

[11] B. Goransson and B. Ottersten, "Direction estimation in partially unknown noise fields," *IEEE Transactions on Signal Processing*, vol. 45, no. 9, pp. 2375–2384, Sept. 1999.

[12] C.E. Chen, F. Lorenzelli, R.E. Hudson, K. Yao, "Stochastic Maximum-Likelihood DOA Estimation in the Presence of Unknown Nonuniform Noise," *IEEE Transactions on Signal Processing*, vol. 56, no. 7, pp. 3038–3044, Jul. 2008.

[13] M. Viberg, P. Stoica, B. Ottersten, "Array processing in correlated noise based on instrumental variables and subspace fitting," *IEEE Transactions on Signal Processing*, vol. 43, pp. 1187–1199, 1995.

[14] P. Palanisamy, N. Kalyanasundaram, A. Raghunandan, "A new DOA estimation algorithm for wideband signals in the presence of unknown spatially correlated noise," *Signal Processing*, vol. 89, pp. 1921–1931, 2009.

[15] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and Cramer-Rao bound," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, no. 5, 720–741, May 1989.

[16] B.D. Rao and K. Hari, "Performance analysis of root-MUSIC," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, 1239–1249, Dec. 1989.

[17] C. Vaidyanathan and K.M. Buckley, "Performance analysis of the MVDR spectrum estimator," *IEEE Trans. Signal Processing*, vol. 44, no. 6, pp. 1427–1437, June 1995.

[18] F. Wen, Q. Wan, H. Wei, F. Rong, "Robust capon beamforming exploiting the second-order noncircularity of signals," *Signal Processing*, 102: 100–111, 2014.

[19] F. Wen, Q. Wan, G. He, P. Zhang, "Wideband direction of arrival estimation with linear array, analysis and closed-form formulae," *Chinese Journal of Electronics*, 22(4): 779–783, 2013.

[20] F. Wen, Q. Wan, L. Y. Luo, "Time-difference-of-arrival estimation for noncircular signals using information theory," *International Journal of Electronics and Communications*, 67(3): 242C245, March 2013.

[21] F. Wen, Q. Wan, "Maximum likelihood and signal-selective TDOA estimation for noncircular signals," *Journal of Communications and Networks*, 15(3): 245-251, June 2013.